small that the exact solution, Eq. (27), has only one root at $Z = 1$. Consequently, the angle of attack variation is the same as that of a critically damped dynamical system as indicated in Fig. 2. The change in reference level ($Y = 0$) is given for several values of k_1. For $Z > 6$ the angle of attack essentially remains zero for the case shown in Fig. 2, and the exact solution given by Eq. (27) for $a = -1$ has no oscillations for $Z > 1$.

It is now evident that for $-a > 10$ Allen's approximate solution, as given by Eq. (12), provides an excellent approximation to his differential equation for the straight line descent of a large drag ballistic missile, Eq. (2). It is necessary to introduce the exact solution, Eq. (7), only when $-a < 10$, a condition which can be attained only by having a sufficiently small C_{m_α} so that $-C_{m_\alpha}(\beta L \sin\theta_E)^{-1} < 10$. In this case the oscillation resembles that of a dynamical system that is near critical damping.

Stone[4] has derived a differential equation similar to Allen's that includes the effect of a constant rate of roll. However, Stone's differential equation for zero roll rate corresponds to $k_3 = 0$ in our Eq. (2), and the omission of C_{L_α} in k_2, Eq. (4). Consequently, Stone's solution for a missile descending without roll would only correspond to the particular solution given by Eq. (27) whenever $C_{L_\alpha} \approx 0$ and $-a > 10$.

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Equivalence of the Minimum Norm and Gradient Projection Constrained Optimization Techniques

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THE title "direct" optimization techniques are shown to be both theoretically and computationally equivalent in the common event that the former is employed as a constrained optimization procedure. The parametric and functional versions of these formulations have heretofore been considered alternative mathematical procedures for iterative solutions of trajectory optimization problems and other problems of similar structure. The reported difference in computational behavior of algorithms based upon the two formulations is traced to inadequate knowledge of the mathematical relationship of the methods and inconsistent computer programming. The results given herein can be employed to eliminate redundancy in existing computer programs and will generally promote increased understanding of these often-used processes.

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Parameter Space Results

Two well-known optimization formulations are studied; these techniques determine values for each of n unknown parameters $\{p_1, p_2, \ldots, p_n\}$ such that a given payoff function

$$\phi = \phi(p_1, p_2, \ldots, p_n) \tag{1}$$

assumes an extreme value, subject to the satisfaction of m equality constraints

$$\psi_j(p_1, p_2, \ldots, p_n) = 0; \quad j = 1, 2, \ldots, m; \quad m < n \tag{2}$$

The functions (1) and (2) are generally nonlinear and typically require numerical integration for their evaluation.

Many numerical techniques have been developed and used in solving a variety of problems of the above type and its function space generalization.[1,2] Among the more popular techniques for solving trajectory optimization problems of the above form are the Gradient Projection and Minimum Norm Optimization techniques. These formulations are based upon local linearizations of the payoff and constraint functions as

$$\Delta\phi = G^T \Delta P \tag{3}$$

and

$$\Delta\Psi - A\Delta P = 0 \tag{4}$$

where $\Delta\phi$ = objective (linearly predicted) charge in the payoff function, $\Delta\Psi = m \times 1$ matrix of constraint violations ("objective minus current"), $G^T = [\partial\phi/\partial p_1 \ldots \partial\phi/\partial p_n]$ = locally evaluated payoff function gradient, $A = [\partial\psi_j/\partial p_i] = m \times n$ locally evaluated constraint Jacobian, and $\Delta P^T = [\Delta p_1 \ldots \Delta p_n]$ = matrix of parameter corrections. The gradient projection formulation determines corrections (ΔP) which locally extremizes the predicted improvement Eq. (3) in ϕ subject to Eq. (4) (first order satisfaction of the constraints) and a restriction upon the norm of the corrections as measured by

$$\Delta s^2 = \Delta P^T W \Delta P \tag{5}$$

where Δs^2 is an assigned value and W is an $n \times n$ positive definite weighting matrix. The minimum norm optimization formulation determines corrections which minimize the control variable norm (Δs^2) required to satisfy the constraints of Eq. (4) and yield a specified (predicted) improvement of Eq. (3) in the payoff function.

Gradient projection Solution

Since this solution, as well as the minimum norm solution, has been fully derived in the literature,[1-5] only the key results are given here. It is the relationship between known results, not derivation of these results, which is being studied. Formally, the correction matrix (ΔP) is sought which extremizes the predicted improvement of Eq. (3) in the payoff function, subject to satisfaction of Eqs. (4) and (5). The desired corrections have been determined[3-5] to be

$$\Delta P = W^{-1}A^T(AW^{-1}A^T)^{-1}\Delta\Psi \pm \left(\frac{\Delta s^2 - \Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi}{G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G}\right)^{1/2} \times [W^{-1}G - W^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G] \tag{6}$$

where the positive (negative) sign has been shown[5] to yield a local maximum (minimum) improvement ($\Delta\phi$). The various conditions under which the scalar numerator and denominator under the square root sign are positive, negative, or zero are given by Junkins[5]; only the most interesting case [when $\Delta s^2 = \Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi$] will be discussed here, after development of the minimum norm solution for comparison.

Minimum norm solution for "constraint-only" case

The minimum norm solution was originally designed for solution of underdetermined boundary constraint problems, with no explicit treatment of the payoff function. As has been accomplished by several investigators,[3-5] however, the minimum norm solution can be readily adapted to solve constrained optimization problems. This is accomplished by restructuring a constrained optimization problem as a sequence of constraint satisfaction problems.

Considering first the "constraint only" version of the minimum norm solution, the objective is to determine the correction matrix $(\Delta\hat{P})$ satisfying the constraint of Eq. (4) which has the minimum norm as measured by Eq. (5). The solution, using the multiplier rule and ordinary calculus, is found to be

$$\Delta\hat{P} = W^{-1}A^T(AW^{-1}A^T)^{-1}\Delta\Psi \tag{7}$$

We note for use in comparison with the gradient projection solution that the norm of Eq. (7) is

$$\Delta\hat{s}^2 = \Delta\hat{P}^T W \Delta\hat{P} = \Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi \tag{8}$$

Upon comparing the minimum norm solution of Eq. (7) with the gradient projection solution of Eq. (6), note that the minimum norm solution is the first term of the gradient projection solution. Note further that one condition under which the second term vanishes is when the correction magnitude (Δs^2) is set equal to the minimum possible $(\Delta\hat{s}^2)$ to satisfy Eq. (4). [In other words, it is meaningless to talk of optimizing $\Delta\phi$ subject to Eq. (4) unless Δs^2 is assigned at least large enough (e.g., $\Delta s^2 \geqq \Delta\hat{s}^2$) to satisfy Eq. (4).] This observation was originally made by Glassman[4] in 1966 and subsequently studied by Junkins[5] in 1969.

Adaptation of the minimum norm solution for constrained optimization

By restructuring the contrained optimization problem as a sequence of constraint problems [by introducing specific improvements $(\Delta\phi)$ as additional "constraint violations" on successive corrections], several investigators have successfully employed the minimum correction formulation to solve constrained optimization problems. Introducing $\Delta\phi$ as the $m+1$th constraint residual and G^T as the $m+1$th row of the Jacobian matrix results in the minimum norm solution assuming the form

$$\Delta P = W^{-1}\begin{pmatrix} A \\ \cdots \\ G^T \end{pmatrix}^T \left[\begin{pmatrix} A \\ \cdots \\ G^T \end{pmatrix} W^{-1} \begin{pmatrix} A \\ \cdots \\ G^T \end{pmatrix}^T\right]^{-1} \begin{pmatrix} \Delta\Psi \\ \cdots \\ \Delta\phi \end{pmatrix} \tag{9}$$

This equation (without the explicit partitioning) has served as the basis of constrained optimization techniques developed by Beskind,[3] Glassman et al.,[4] and Junkins[5]; with $\Delta\phi$ being adjusted before each iteration in an analogous fashion to the required adjustment of Δs^2 in the gradient projection solution of Eq. (6). In the following section, we will show that if $\Delta\phi$ is assigned consistent with Δs^2 of the gradient projection solution that the minimum norm optimization solution of Eq. (9) is simply another form of Eq. (6). It will then be clear that any difference in computational behavior of the two techniques must stem from inconsistent procedures for assigning $\Delta\phi$ in Eq. (9) and Δs^2 in Eq. (6).

Equivalence of the minimum norm and gradient projection formulations

The minimum norm optimization solution can be written as

$$\Delta P = [W^{-1}A^T \,\vdots\, W^{-1}G] \begin{pmatrix} AW^{-1}A^T & \vdots & AW^{-1}G \\ \cdots & & \cdots \\ G^TW^{-1}A^T & \vdots & G^TW^{-1}G \end{pmatrix}^{-1} \begin{pmatrix} \Delta\Psi \\ \cdots \\ \Delta\phi \end{pmatrix} \tag{10}$$

Following the standard techniques[6] for inverting properly partitioned† matrices, Eq. (10) can be expanded as

$$\Delta P = W^{-1}A^TC_{11}\Delta\Psi + W^{-1}A^TC_{12}\Delta\phi + W^{-1}GC_{21}\Delta\Psi + W^{-1}GC_{22}\Delta\phi \tag{11}$$

where the submatrices of the inverse are

$$C_{11} = [AW^{-1}A^T - AW^{-1}G(G^TW^{-1}G)^{-1}G^TW^{-1}A^T]^{-1} \tag{12a}$$

$$C_{22} = [G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \tag{12b}$$

$$C_{12} = -[G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \times (AW^{-1}A^T)^{-1}AW^{-1}G \tag{12c}$$

$$C_{21} = -(G^TW^{-1}G)^{-1}G^TW^{-1}A^T \times [AW^{-1}A^T - AW^{-1}G(G^TW^{-1}G)^{-1}G^TW^{-1}A^T]^{-1} \tag{12d}$$

† Diagonal submatrices must be square and nonsingular.

Due to symmetry of the matrix (10) before inversion, then

$$C_{12} = C_{21}{}^T \tag{13}$$

from which we obtain the useful identities

$$(G^TW^{-1}G)^{-1}[AW^{-1}A^T - AW^{-1}G(G^TW^{-1}G)^{-1}G^TW^{-1}A^T]^{-1} = [G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \times (AW^{-1}A^T)^{-1} \tag{14}$$

and, inverting both sides of Eq. (14),

$$AW^{-1}GG^TW^{-1}A^T = [G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]AW^{-1}A^T \tag{15}$$

Making use of Eq. (14), the submatrices (12) can then be written in the final forms

$$C_{11} = (G^TW^{-1}G)[G^TW^{-1}G - G^TW^{-1}A^T \times (AW^{-1}A^T)^{-1}AW^{-1}G]^{-1}(AW^{-1}A^T)^{-1} \tag{16a}$$

$$C_{12} = -[G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \times (AW^{-1}A^T)^{-1}AW^{-1}G \tag{16b}$$

$$C_{21} = -[G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \times G^TW^{-1}A^T(AW^{-1}A^T)^{-1} \tag{16c}$$

$$C_{22} = [G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \tag{16d}$$

Substitution of Eq. (16) into Eq. (11) yields a final form for the expanded minimum norm optimization solution as

$$\Delta P = [G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \times \{(G^TW^{-1}G)A^T(AW^{-1}A^T)^{-1} - A^T(AW^{-1}A^T)^{-1} \times AW^{-1}G\Delta\phi - GG^TW^{-1}A(AW^{-1}A^T)^{-1}\Delta\Psi + G\Delta\Psi\} \tag{17}$$

An expression relating the assigned $\Delta\phi$ to the corresponding Δs^2 can be found by substituting Eq. (17) into Eq. (5); this yields the quadratic

$$\Delta s^2 = [G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G]^{-1} \times \{\Delta\phi^2 - 2[G^TW^{-1}A^T(AW^{-1}A^T)^{-1}\Delta\Psi]\Delta\phi + (G^TW^{-1}G)\Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi\} \tag{18}$$

Equation (18) can be solved for the assigned $\Delta\phi$ in terms of the corresponding Δs^2 as

$$\Delta\phi = G^TW^{-1}A^T(AW^{-1}A^T)^{-1}\Delta\Psi \pm [G^TW^{-1}G - G^TW^{-1}A^T(AW^{-1}A^T)^{-1}AW^{-1}G] \times [\Delta s^2 - \Delta\Psi^T(AW^{-1}A^T)^{-1}\Delta\Psi]^{1/2} \tag{19}$$

The equivalence of the minimum norm and gradient projection formulations is now established in a straightforward manner by substituting Eq. (19) into the expanded minimum norm solution of Eq. (17) (to eliminate $\Delta\phi$ in terms of the corresponding Δs^2); after some algebra, one obtains the gradient projection solution of Eq. (6). Thus, the formulations differ only in whether Δs^2 or $\Delta\phi$ must be assigned, and if they are assigned consistently according to Eq. (18) or Eq. (19); then the corrections obtained are identical. Equation (18) or (19) can be immediately employed to convert a given minimum norm optimization algorithm into the equivalent gradient projection algorithm, or vice versa.

Function Space Results

For brevity, and to prevent obscuring the basic results, only the parameter space results have been given here. These results have been fully generalized[7] to establish the equivalence of the gradient projection and minimum norm algorithms in function space.

Conclusions

It has been shown that the often-used optimization adaptation of the minimum norm formulation is mathematically equivalent to the gradient projection formulation. The reported differences[3–5] in computational behavior of algorithms based upon the two solutions is clearly due to inconsistencies in the logic required to assign $\Delta\phi$ in the former and Δs^2 in the latter. Since the optimum procedure for controlling either is a 'will-o'-the wisp" and can never be found, it is not surprising that independent attempts at controlling them has not yielded equivalent algorithms. The results given herein and in Ref. 7 can be immediately employed to transform a given gradient projection algorithm into the equivalent minimum norm algorithm, and vice versa. These results therefore can be used to eliminate redundancy in existing computer programs and will allow improved understanding of these often-used techniques.

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Flow Pattern of Two Impinging Circular Jets

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THIS Note describes a method to determine the flow pattern resulting from the impingement of two circular liquid streams. Using finite-difference techniques, values of the stream function are calculated and the velocity and pressure distributions along the impingement surface are obtained.

Figure 1 shows two circular jets impinging on each other. The centerlines of the two jets coincide so that the flow is axially symmetric. The flow is assumed to be inviscid, incompressible and irrotational, and no mixing of the two streams occurs. Because of symmetry, the stream function, ψ, can be defined by

$$u = (1/r)\,\partial\psi/\partial z \tag{1}$$

and

$$w = -(1/r)\,\partial\psi/\partial r \tag{2}$$

where r is the radial coordinate, z is the axial coordinate, and u and w are the radial and axial components of fluid velocity, respectively. The equation of irrotationality in axisymmetric flow is

$$\partial u/\partial z - \partial w/\partial r = 0 \tag{3}$$

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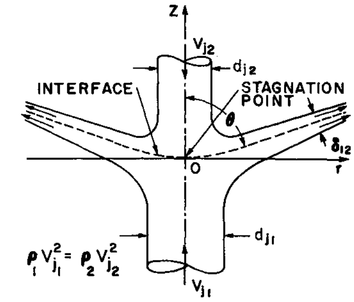

Fig. 1 Impingement of unequal, opposite, circular jets.

Substituting Eqs. (1) and (2) into Eq. (3) yields the following equation for the stream function:

$$\partial^2\psi/\partial r^2 + \partial^2\psi/\partial z^2 - (1/r)\,\partial\psi/\partial r = 0 \tag{4}$$

A necessary condition for a meaningful problem is that the stagnation pressures of the two jets be equal. Otherwise, the jet of higher stagnation pressure would drive the other jet back to its source. The stagnation pressure is given by

$$p_s = p_a + \rho V_j^2/2g \tag{5}$$

where p_s is the stagnation pressure, p_a is the ambient pressure, ρ is the fluid density, and V_j is the upstream jet velocity. It is assumed that the two jets have equal density and initial velocity, thereby satisfying the requirement for equal stagnation pressures. The velocity and pressure distributions at the interface are found from the steady flow Bernoulli's equation

$$p_s = p + \rho u^2/2g \tag{6}$$

The static pressure p must be continuous across the impingement surface, and therefore the velocity must also be continuous.

The boundary conditions applied to the model are as follows:[1] 1) The fluid velocity V_j along the jets' free surface is constant and in view of Eqs. (1) and (2), the gradient of ψ at the free surface and normal to it, $\partial\psi/\partial n$, is equal to $V_j r$. 2) The stream function ψ along the centerline of each jet and on the impingement surface is a constant and is set equal to zero. 3) It is assumed that the fluid velocity V_j across the boundaries of the incoming and outgoing flow is constant. Therefore, the stream function ψ at the upstream boundary is equal to $\frac{1}{2}V_j r^2$, and the stream function at the outgoing boundary is equal to $V_j rz$ $(\sin\theta)$, where z is the axial distance from the impingement surface and θ is the angle between the jets' axis and the final flow direction of the outgoing flow. 4) The stream function ψ on the free surface is constant and, according to conditions 2 and 3, is equal to $\frac{1}{2}V_j r_j^2$, where r_j is the upstream jet radius.

The boundary for the incoming flow is established at an axial distance $3r_j$ from the stagnation point. The boundary for the outgoing fluid sheet is at a radial distance $4r_j$ from the stagnation point. These boundaries are based on experimental results[2] and represent the locations where the jet flow begins to deviate due to impingement. Since the downstream boundary is chosen at $4r_j$, then ψ on this boundary is $4V_j r_j z(\sin\theta)$.

Method of Solution

By assuming the surface contours of the two jets and the contour of the impingement surface, and a set of boundary conditions on these surfaces, a solution to Eq. (4) can be obtained for each jet. A second boundary condition on each of the assumed surfaces is then applied to determine whether the assumed contours are correct. The solution for the jet flow is obtained when the jets' boundaries satisfy both sets of boundary conditions. The first boundary condition at each free surface is that ψ is constant and equal to $\frac{1}{2}V_j r_j^2$. The second boundary condition is $\partial\psi/\partial n = V_j r$. At the impingement surface, the first and second boundary conditions are, respectively, $\psi = 0$ and the pressure distribution of the two jets match. Solutions are obtained for each jet separately, assuming an interface contour and calculating the free surface contour which is correct for that interface. If the interface pressure distributions do not match, the interface contour is modified and the procedure is repeated.